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STABLE STRUCTURE AND CANCELLATION OF QUADRATIC SPACES OVER LAURENT EXTENSIONS OF RINGS OF DIMENSION ONE

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1. Introduction

Let R be a commutative Noetherian ring in which 2 is invertible. In [3] Karoubi has proved that if R is a regular ring, then $W(R[T,T^{-1}]) \xrightarrow{\sim} W(R) \oplus W(R)$ where W denotes the Witt ring functor. In this paper we show that if R is a ring of dimension one with finite normalisation $\bar{R}$, then for any quadratic space q over $R[T,T^{-1}]$ there exist quadratic spaces q_0, q_1 over R such that $[q]=[q_0 \perp Tq_1]$, $[\cdot]$ denoting the equivalence class in $W(R[T,T^{-1}])$. Using this, in Theorem 2.4 we prove that

$$0 \to K \to W(R) \oplus W(R) \to W(R[T,T^{-1}]) \to 0$$

is an exact sequence of groups where K is the kernel of the canonical map $W(R) \to W(\bar{R}) \oplus W(R/\mathfrak{C})$, $\mathfrak{C}$ being the conductor of R in $\bar{R}$. We also prove (Theorem 3.2) that quadratic spaces over $R[T,T^{-1}]$ of Witt index ≥ 2 are cancellative. This is an improvement of the general cancellation theorem [8, Theorem 7.2] for this particular case. The proof of these results uses the structure of the orthogonal group of isotropic quadratic spaces over $k[T,T^{-1}]$, where k is a field, which is given in Lemma 1.3.

In this paper we assume that 2 is invertible in all rings considered. Also for any ring R, by $\mathscr{P}(R)$ we will mean the class of all finitely generated projective R-modules and $\mu_2(R)=\{x \in R \mid x^2=1\}$.

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1. Orthogonal transformations

In this section, we include a few lemmas which are needed in this paper.

Let R be a commutative ring. Let (Q,q) be a quadratic space over R, h be the hyperbolic plane $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ and let $Re \oplus Rf$ be the underlying module of the form h with

$\langle e,f\rangle=1$, $\langle e,e\rangle=0=\langle f,f\rangle$. For $w\in Q$ the elements $E_w, E_w^*\in O(q\perp h)$ are defined as follows [8, p. 291]:

$$E_w(z)=z+\langle z,w\rangle e,\quad z\in Q,$$

$$E_w(e)=e,\qquad E_w(f)=-w-q(w)e+f$$

and

$$E_w^*(z)=z+\langle z,w\rangle f,\quad z\in Q,$$

$$E_w^*(e)=-w-q(w)f+e,\qquad E_w^*(f)=f.$$

Let $EO_R(q,h)$ denote the subgroup of $O_R(q\perp h)$ generated by the set $\{E_w, E_w^* \mid w\in Q\}$. Recall [4] that $O_R(h)$ normalises $EO_R(q,h)$. Let $G_R(q,h)$ denote the subgroup $EO_R(q,h)\cdot O_R(h)$ of $O_R(q\perp h)$. Let $U(R)$ denote the set of units of R. For any u in $U(R)$ let τ_u denote the element $\begin{pmatrix} u & 0\\ 0 & u^{-1}\end{pmatrix}$ of $O_R(h)$.

Lemma 1.1. *Let k be a field of characteristic $\neq 2$ and q be a quadratic space over k. If, for $a\in k^*$, $\tau_a\in EO_k(q,h)$, then $a=b^2$ where $b\in k^*$.*

Proof. See [2, p. 27, Theorem 4.6].

Lemma 1.2. *Let R be a domain in which 2 is a unit. Let (Q,q) be a quadratic space over R which represents a unit. Then, for any unit u in R, $\mathrm{Id}\perp\tau_{u^2}\in EO_R(q,h)$.*

Proof. Let $w\in Q$ such that $q(w)\in U(R)$. Let $s\in R$ such that $1-sq(w)=u^{-1}$. Then, we have [2, p. 16, 3.16]

$$\tau_{u^2}=E^*_{-suw^{-1}}E_{-uw}E^*_{sw}E_w.$$

Thus $\tau_{u^2}\in EO_R(q,h)$.

Lemma 1.3. *Let k be a field of characteristic $\neq 2$ and (Q,q) a quadratic space over $R=k[X,X^{-1}]$. Then $O_R(q\perp h)=G_R(q,h)$.*

Proof. The proof is by induction on rank q. If rank $q=0$, then $O_R(q\perp h)=O_R(h)$. We assume rank $q=n>0$. By [5, Lemma 1.2] Q has an orthogonal basis $\{e_1,\dots,e_n\}$ with $q(e_i)=\lambda_i$ where $\lambda_i\in k^*$ or $\lambda_i=\mu_iX$, $\mu_i\in k^*$. As in the proof of [6, Lemma 1.1] it follows that $O_R(q)\subseteq G_R(q,h)$. Let $\alpha\in O_R(q\perp h)$ with $\alpha(f)=\xi+ae+bf$, where $\xi=\sum_i a_ie_i$, $a_i,a,b\in R$, $\langle e,e\rangle=0=\langle f,f\rangle$ and $\langle e,f\rangle=1$. In case $\xi=0$ or $\xi\neq 0$ and a or b is a unit in R, we see that $\alpha\in G_R(q,h)$, as in the proof of [6, Lemma 1.1]. Suppose that $\xi\neq 0$ and neither a nor b is a unit. We induct on $\min(d(a),d(b))$ where d denotes the Euclidean function on R induced by the degree function on $k[X]$. Consider the ideal bR. There exists $p\in k[X]$, with $1+Xp\in bR$. Let $b'\in R$ be such that $1+Xp=bb'$. For each $i=1,\dots,n$ there are $g_i\in k[X]$ and $c_i\in R$ with $a_i=g_i+c_i(1+Xp)=g_i+c_ibb'$. Then

$$E_{\sum_i b'c_ie_i}\circ\alpha(f)=\sum_i g_ie_i+\left(a-b'\sum_i a_ic_i\lambda_i-\tfrac{1}{2}bb'^2\sum_i c_i^2\lambda_i\right)e+bf$$

and $g_i \in k[X]$ for all $i=1,\dots,n$. Thus, we may assume

$$\alpha(f)=\sum_i a_ie_i+ae+bf \quad \text{and} \quad a_i\in k[X] \quad \text{for } i=1,\dots,n.$$

Assume $d(a)\le d(b)$, the proof being similar if $d(b)\le d(a)$. If $a=a'X^s$ with $(a',X)=1$ and $a'\in k[X]$, then

$$\tau_{X^{s'}}\circ\alpha(f)=\sum_i a_ie_i+a'X^{s+s'}e+bX^{-s'}f$$

and $d(a)=\deg a'=d(a'X^{s+s'})$. Therefore we may also assume that $\alpha(f)=\sum_i a_ie_i+ae+bf$ with $a_i\in k[X]$, $a=a'X^l\in k[X]$, $l\gg 0$, $(a',X)=1$, $a'\in k[X]$. For each $i=1,\dots,n$, given a_i and a' there are l_i and m_i in $k[X]$ with $a_i=a'l_i+m_i$, $m_i=0$ or $\deg m_i<\deg a'=d(a)$. Thus $a_i=aq_i+m_i$ with $q_i\in R$ and $m_i\in k[X]$, $d(m_i)<d(a)$. Let $w=-\sum_i q_ie_i$. Then

$$E^*_{-w}\circ\alpha(f)=(\xi+aw)+ae+b''f \quad \text{where } b''=b-aq(w)-\langle\xi,w\rangle.$$

Since $q(\alpha(f))=0$, we have $\sum_i a_i^2\lambda_i+2ab=0$. Hence

$$\sum_i m_i^2\lambda_i=-a'X^l\Big(2b+2\sum_i q_im_i\lambda_i+\sum_i aq_i^2\lambda_i\Big).$$

We choose $l\gg 0$ so that $X\mid\sum_i m_i^2\lambda_i$. If $\max_i\{\deg m_i^2\lambda_i\}$ is attained at j, we have

$$d\Big(\sum_i m_i^2\lambda_i\Big)\le\deg\Big(\sum_i m_i^2\lambda_i\Big)-1\le\deg(m_j^2\lambda_j)-1$$
$$\le 2\deg m_j<2\deg a'=2d(a).$$

Thus $d(\sum_i m_i^2\lambda_i)<2\,d(a)$ and $d(q(\xi+aw))<2\,d(a)$. Since $q(E^*_{-w}\alpha(f))=0$, we have $q(\xi+aw)=-2ab''$ and $d(b'')<d(a)$. Thus we have reduced $\min(d(a),d(b))$ by applying elements of $G_R(q,h)$ to α. During this procedure the first n components of $\alpha(f)$ remain polynomials, and hence, repeating the above process only involves applying elements of $O_R(h)$ and so we continue decreasing $\min(d(a),d(b))$ till it becomes zero.

Remark 1.4. Let k be a field with characteristic different from 2 and q be a diagonisable space over $R=k[X,1/f]$, $f\in k[X]$. Then it can be similarly proved that $O_R(q\perp h)=G_R(q,h)$.

2. Stable structure of quadratic spaces over $R[T,T^{-1}]$

Let R be a reduced ring with finite normalisation $\bar{R}$ and let $\mathfrak{C}$ be the conductor of R in $\bar{R}$. Let $A,\bar{A},A/\mathfrak{C},\bar{A}/\mathfrak{C}$ denote the Laurent extensions $R[T,T^{-1}]$, $\bar{R}[T,T^{-1}]$, $(R/\mathfrak{C})[T,T^{-1}]$, $(\bar{R}/\mathfrak{C})[T,T^{-1}]$ respectively. Then we have the Cartesian squares

$$(1)\quad \begin{array}{ccc} R & \longrightarrow & \bar{R} \\ \downarrow & & \downarrow \\ R/\mathfrak{C} & \longrightarrow & \bar{R}/\mathfrak{C} \end{array} \qquad \text{and} \qquad (2)\quad \begin{array}{ccc} A & \longrightarrow & \bar{A} \\ \downarrow & & \downarrow \\ A/\mathfrak{C} & \longrightarrow & \bar{A}/\mathfrak{C} \end{array}$$

By '$\sim$' we will denote 'going modulo $\langle T-1\rangle$'.

Lemma 2.1. *Let S be a ring in which 2 is invertible. Then*

$$\mu_2(S[T,T^{-1}])=\mu_2(S).$$

Proof. Let $f=T^{-s}(a_0+a_1T+\cdots+a_rT^r)\in\mu_2(S[T,T^{-1}])$. Suppose first that S is reduced. The equation $f^2=1$ gives $f=a_r\in\mu_2(S)$. Now let S be arbitrary and let '′' denote 'going modulo N' where N is the nil radical of S. Then $f'\in\mu_2(S'[T,T^{-1}])=\mu_2(S')$. Let $a\in U(S)$ such that $a'=f'$. Then $a^2=(1+y)^2$ for some $y\in N$, since $2\in U(S)$. Thus $[fa^{-1}(1+y)]'=1'$ and $[fa^{-1}(1+y)]^2=1$ so that there exists $z\in N$ with $fa^{-1}(1+y)=1+z$. The equation $(1+z)^2=1$ implies $z=0$. Hence $f\in\mu_2(S)$.

Lemma 2.2. *With notation as above, let q be a quadratic space over A such that $q\otimes_A\bar{A}$ and $q\otimes_A A/\mathfrak{C}$ are extended from $\bar{R}$ and $R/\mathfrak{C}$ respectively. Then q is stably extended from R.*

Proof. Let $\varphi: q\otimes_A\bar{A}\xrightarrow{\sim}\tilde{q}\otimes_A\bar{A}$ and $\psi: q\otimes_A A/\mathfrak{C}\xrightarrow{\sim}\tilde{q}\otimes_A A/\mathfrak{C}$ be isometries over $\bar{A}$ and $A/\mathfrak{C}$ respectively such that $\tilde{\varphi}=\tilde{\psi}=\mathrm{Id}$. Then $\varphi^{*-1}\psi^*\in O_{\bar{A}/\mathfrak{C}}(q)$, where '$*$' denotes the extensions to $\bar{A}/\mathfrak{C}$. Hence $\varphi^{*-1}\psi^*\perp\mathrm{Id}\in O_{\bar{A}/\mathfrak{C}}(q\perp h)$. Since $\bar{R}/\mathfrak{C}$ is a product of fields modulo its radical, by Lemma 1.3 there exist $\eta\in EO_{\bar{A}/\mathfrak{C}}(q,h)$ and $\tau\in O_{\bar{A}/\mathfrak{C}}(h)$ such that $\varphi^{*-1}\psi^*\perp\mathrm{Id}=\eta\tau$. Since $\tilde{\eta}\tilde{\tau}=\mathrm{Id}$ and $\det\tilde{\eta}=1$ we have $\det\tilde{\tau}=1$. Since $\mu_2(\bar{A}/\mathfrak{C})=\mu_2(\bar{R}/\mathfrak{C})$, we have $\det\tau=\det\tilde{\tau}=1$. Thus $\tau=\tau_u$ for some $u\in U(\bar{A}/\mathfrak{C})$. Using the Cartesian square (2) and Milnor's result [1, Ch. IX, 5.1] we obtain a rank-1 projective A-module P corresponding to the triple $(\bar{A},u,A/\mathfrak{C})$. Since $\bar{A}\to\bar{A}/\mathfrak{C}$ is surjective we can lift η to $EO_{\bar{A}}(q,h)$ and alter φ suitably to assume that $\varphi^{*-1}\psi^*\perp\mathrm{Id}=\tau_u$. Thus the triples $(q\perp h,\mathrm{Id}\perp\tau_u,q\perp h)$ and $(\tilde{q}\perp h,\mathrm{Id},\tilde{q}\perp h)$ are equivalent and hence $q\perp H(P)\xrightarrow{\sim}\tilde{q}\perp h$.

We now analyse the structure of the Witt ring $W(R[T,T^{-1}])$. For this we consider the group $K=\mathrm{Ker}(W(R)\to W(\bar{R})\oplus W(R/\mathfrak{C}))$, the map being the diagonal map induced by the natural maps $R\to\bar{R}$ and $R\to R/\mathfrak{C}$. The class of any quadratic space in the Witt ring will be denoted by $[\cdot]$. We first remark that given $[q]\in K$, there are isometries

$$\varphi:(q\perp h)\otimes_R\bar{R}\xrightarrow{\sim}(H(P)\perp h)\otimes_R\bar{R}$$

over $\bar{R}$ and

$$\psi:(q\perp h)\otimes_R R/\mathfrak{C}\xrightarrow{\sim}(H(P)\perp h)\otimes_R R/\mathfrak{C}$$

over $R/\mathfrak{C}$ for some $P\in\mathscr{P}(R)$. Then $\varphi^*\psi^{*-1}\in O_{\bar{R}/\mathfrak{C}}(H(P)\perp h)$, '$*$' denoting the extensions to $\bar{R}/\mathfrak{C}$. Since $\bar{R}/\mathfrak{C}$ is a product of fields modulo its radical, by [2, Theorem 3.3] there exist $\eta\in EO_{\bar{R}/\mathfrak{C}}(H(P),h)$ and $\tau\in O_{\bar{R}/\mathfrak{C}}(h)$ such that $\varphi^*\psi^{*-1}=\eta\tau$. We can lift η to an element of $EO_{\bar{R}}(H(P),h)$ and alter φ suitably to assume that $\varphi^*\psi^{*-1}=\tau$. Using the Cartesian square (1) we get a rank 2 quadratic space q_0 over R which corresponds to the triple (h,τ,h). Thus $q\perp h\xrightarrow{\sim}H(P)\perp q_0$ and hence, $[q]=[q_0]$ in $W(R)$. *Thus any element of K has a representative of the form* $[q_0]$. The trivial element of K corresponds to the class of (h,τ_u,h) for any $u\in U(\bar{R}/\mathfrak{C})$.

Using this representation we now prove

Proposition 2.3. *With the notation as above, the map $F:K\to\mu_2(\bar{R}/\mathfrak{C})/(\mu_2(\bar{R})\cdot\mu_2(R/\mathfrak{C}))$ defined by $F([q])=\overline{\det\tau}$ is a group isomorphism, where q corresponds to the triple (h,τ,h) and bar denotes the class in $\mu_2(\bar{R}/\mathfrak{C})(\mu_2(\bar{R})\cdot(\mu_2(R/\mathfrak{C}))$.*

Proof. F is well-defined, since τ uniquely defines $[q]$ up to equivalence. Let $[q_0],[q_0']\in K$ be defined by the class of the triples (h,τ,h) and (h,τ',h). Then $\tau\perp\tau'\in O_{\bar{R}/\mathfrak{C}}(h\perp h)$ and hence there exist $\eta''\in EO_{\bar{R}/\mathfrak{C}}(h,h)$ and $\tau''\in O_{\bar{R}/\mathfrak{C}}(h)$ such that $\tau\perp\tau'=\eta''\tau''$. Then

$$\begin{aligned}F([q_0]+[q_0'])&=F([q_0\perp q_0'])=\overline{\det\tau''}\\&=\overline{\det\tau}\cdot\overline{\det\tau'}=F([q_0])\cdot F([q_0'])\end{aligned}$$

which shows that F is a homomorphism.

Let $[q]\in K$ be given by the class of the triple (h,τ,h) and $F([q])=1$. Then $\det\tau=uv$ for $u\in\mu_2(\bar{R})$, $v\in\mu_2(R/\mathfrak{C})$. Since 2 is a unit in $\bar{R}$ and $R/\mathfrak{C}$, the maps $O_{\bar{R}}(h)\xrightarrow{\det}\mu_2(\bar{R})$ and $O_{R/\mathfrak{C}}(h)\xrightarrow{\det}\mu_2(R/\mathfrak{C})$ are surjective. Thus we can lift u and v and alter τ suitably to assume that (h,τ,h) is equivalent to $(h,\tau_{u'},h)$ for some $u'\in\mu_2(\bar{R}/\mathfrak{C})$. Thus $[q]=0$, showing that F is injective.

Let $u\in\mu_2(\bar{R}/\mathfrak{C})$. Let $\tau\in O_{\bar{R}/\mathfrak{C}}(h)$ s.t. $\det\tau=u$. Then (h,τ,h) defines an element $[q]\in K$ such that $F([q])=\bar{u}$ and hence F is surjective.

We next compute K for two rings.

Example 1 [7, Proposition 4.5]. $R=k[t^2,t^3]$, k is a field of characteristic $\neq 2$. Then $\bar{R}=k[t]$, $I=\langle t^2,t^3\rangle$, $R/\mathfrak{C}\xrightarrow{\sim}k$, $\bar{R}/\mathfrak{C}\xrightarrow{\sim}k[t]/\langle t^2\rangle$, $\mu_2(R/\mathfrak{C})=\{\pm1\}=\mu_2(\bar{R}/\mathfrak{C})=\mu_2(\bar{R})$. Thus $K\xrightarrow{\sim}\{1\}$.

Example 2. Let $R=\mathbb{C}[X,Y]/\langle Y^2-X^2-X^3\rangle$. Then $\bar{R}=\mathbb{C}[y/x]$ where x and y denote the classes of X and Y modulo $\langle Y^2-X^2-X^3\rangle$ and $\mathfrak{C}=\langle x,y\rangle$. Thus $R/\mathfrak{C}\xrightarrow{\sim}\mathbb{C}$ and $\bar{R}/\mathfrak{C}\xrightarrow{\sim}\mathbb{C}[y/x]/\langle(y/x)^2-1\rangle$ so that $\mu_2(\bar{R})=\mu_2(R/\mathfrak{C})=\{\pm1\}$ and $\mu_2(\bar{R}/\mathfrak{C})=\{\pm1,\pm y/x\}$. Thus $K\xrightarrow{\sim}\mathbb{Z}/2\mathbb{Z}$.

We now use K to prove the following

Theorem 2.4. *There is an exact sequence of groups*

$$0 \to K \xrightarrow{f} W(R) \oplus W(R) \xrightarrow{g} W(R[T,T^{-1}]) \to 0$$

where $f([q]) = ([q],[q])$ *and* $g([q_1],[q_2]) = [q_1] - [Tq_2]$ *for* $[q] \in K$ *and* $[q_1],[q_2] \in W(R)$.

Proof. Clearly f and g are well-defined and f is injective. To prove that g is surjective we consider $[q] \in W(R[T,T^{-1}])$ where the representative q is of rank ≥ 3 and has Witt index ≥ 1. Since $\bar{R}$ is a product of Dedekind domains by [5, Theorem 3.5] we have $[q \otimes_A \bar{A}] = [q_0' \perp Tq_1']$ in $W(\bar{A})$ for some quadratic spaces q_0' and q_1' over $\bar{R}$. Since $R/\mathfrak{C}$ is a product of fields modulo its radical, by [5, Lemma 1.2] there exist quadratic spaces q_0'', q_1'' over $R/\mathfrak{C}$ such that $[q \otimes_A A/\mathfrak{C}] = [q_0'' \perp Tq_1'']$ in $W(A/\mathfrak{C})$. Thus

$$[(q_0' \perp Tq_1') \otimes_{\bar{A}} \bar{A}/\mathfrak{C}] = [(q_0'' \perp Tq_1'') \otimes_{A/\mathfrak{C}} A/\mathfrak{C}]$$

in $W(\bar{A}/\mathfrak{C})$. Now there exist integers $l_0, l_1, m_0, m_1 \geq 0$ and anisotropic quadratic spaces $q_0^{*\prime}, q_0^{*\prime\prime}, q_1^{*\prime}, q_1^{*\prime\prime}$ over $\bar{R}/\mathfrak{C}$ such that

$$q_0' \otimes_{\bar{A}} \bar{A}/\mathfrak{C} \xrightarrow{\sim} q_0^{*\prime} \perp h^{l_0}, \qquad q_0'' \otimes_{A/\mathfrak{C}} \bar{A}/\mathfrak{C} \xrightarrow{\sim} q_0^{*\prime\prime} \perp h^{m_0},$$

$$q_i' \otimes_{\bar{A}} \bar{A}/\mathfrak{C} \xrightarrow{\sim} q_1^{*\prime} \perp h^{l_1}, \qquad q_1'' \otimes_{A/\mathfrak{C}} \bar{A}/\mathfrak{C} \xrightarrow{\sim} q_1^{*\prime\prime} \perp h^{m_1}.$$

Since $\bar{R}/\mathfrak{C}$ is a product of fields modulo its radical, by [5, Lemma 1.3] we have $q_0^{*\prime} \perp Tq_1^{*\prime} \xrightarrow{\sim} q_0^{*\prime\prime} \perp Tq_1^{*\prime\prime}$. Thus $q_0^{*\prime} \xrightarrow{\sim} q_0^{*\prime\prime}$ and $q_1^{*\prime} \xrightarrow{\sim} q_1^{*\prime\prime}$ over $\bar{R}/\mathfrak{C}$. Hence there are integers $i_0, i_1, j_0, j_1 \geq 0$ and isometries

$$\beta_0 : (q_0' \perp h^{i_0}) \otimes_{\bar{R}} \bar{R}/\mathfrak{C} \xrightarrow{\sim} (q_0'' \perp h^{j_0}) \otimes_{R/\mathfrak{C}} \bar{R}/\mathfrak{C}$$

and

$$\beta_1 : (q_1' \perp h^{i_1}) \otimes_{\bar{R}} \bar{R}/\mathfrak{C} \xrightarrow{\sim} (q_1'' \perp h^{j_1}) \otimes_{R/\mathfrak{C}} \bar{R}/\mathfrak{C}$$

over $\bar{R}/\mathfrak{C}$. Let q_0, q_1 be quadratic spaces over R defined by the triples $(q_0' \perp h^{i_0}, \beta_0, q_0'' \perp h^{j_0})$ and $(q_1' \perp h^{i_1}, \beta_1, q_1'' \perp h^{j_1})$ respectively and obtained from the Cartesian square (1). Let $[P] = [q] - [q_0] - [Tq_1]$ in $W(R[T,T^{-1}])$. Then $[p]$ is trivial in $W(\bar{A})$ and $W(A/\mathfrak{C})$. Thus $p \otimes \bar{A}$ and $p \otimes A/\mathfrak{C}$ are extended from $\bar{R}$ and $R/\mathfrak{C}$ respectively. Hence, by Lemma 2.2, p is stably extended from R. Thus $[p] = [\tilde{p}] \in W(R)$. Thus

$$[q] = [q_0 \perp \tilde{p} \perp Tq_1] = g([q_0 \perp \tilde{p}], [-q_1]).$$

To show that $\operatorname{Ker} g = \operatorname{im} f$ consider $([q_1],[q_2]) \in \operatorname{Ker} g$. Then $[q_1] = [Tq_2]$ and hence $[q_1] = [\tilde{q}_1] = [\tilde{T}q_2] = [q_2] = [q]$, say, in $W(R)$. Since $\bar{R}$ is a product of Dedekind domains, we may assume $\bar{R}$ is a domain with quotient field k. Then $[q] = [Tq]$ in $W(k[T,T^{-1}])$ so that $q \perp h^r \xrightarrow{\sim} Tq \perp h^r$ over $k[T,T^{-1}]$ for some $r > 0$. Thus, by [5, Lemma 1.3], $q \xrightarrow{\sim} Tq$. Hence q is hyperbolic over k. Since $W(\bar{R}) \to W(k)$ is injective, $q \otimes_R \bar{R}$ is hyperbolic over $\bar{R}$. Similarly, since $[q] = [Tq]$ over $A/\mathfrak{C}$ and $R/\mathfrak{C}$ is a product of fields modulo its radical, $q \otimes_R R/\mathfrak{C}$ is hyperbolic over $R/\mathfrak{C}$.

Thus $[q[\in K$ and $f([q])=([q_1],[q_2])$, so that $\operatorname{Ker} g \subseteq \operatorname{Im} f$. Also, if $[q]\in K$, then q is stably hyperbolic over $\bar{R}$ and $R/\mathfrak{C}$, so that Tq is stably hyperbolic over $\bar{A}$ and $A/\mathfrak{C}$. Thus, by Lemma 2.2, $[Tq]=[\tilde{T}q]=[q]$ and hence $gf([q])=[q]-[Tq]=0$, so that $\operatorname{Im} f \subseteq \operatorname{Ker} g$.

Remark. In the course of the proof of the above theorem we have proved that given a quadratic space q over $R[T,T^{-1}]$ there exist quadratic spaces q_0 and q_1 over R such that $[q]=[q_0 \perp Tq_1]$ in $W(R[\dot{T},T^{-1}])$.

3. Cancellation of quadratic spaces over $R[T,T^{-1}]$

Proposition 3.1. *Let R be a Dedekind domain and q be a quadratic space over* $R[T,T^{-1}]$ *of Witt index* ≥ 2. *Then q is cancellative.*

Proof. It is enough to prove that if q' is a quadratic space over $R[T,T^{-1}]$ such that $q \perp h \xrightarrow{\sim} q' \perp h$, then $q \xrightarrow{\sim} q'$. By [5, Theorem 3.5], $q \xrightarrow{\sim} q_1 \perp Tq_2 \perp h \perp H(Q)$ where q_1 and q_2 are quadratic spaces over R and Q is a rank-1 projective R-module. Thus q' is stably isometric to $(q_1 \perp h) \perp Tq_2 \perp H(Q)$ so that $q' \xrightarrow{\sim} (q_1 \perp h) \perp Tq_2 \perp H(Q')$ where Q' is a rank-1 projective R-module [5, proof of Theorem 3.5]. Since Witt index $\tilde{q} \geq 2$ and $\dim R=1$ and

$$q_1 \perp q_2 \perp h \perp H(Q') \perp h \xrightarrow{\sim} q_1 \perp q_2 \perp \dot{h} \perp H(Q) \perp h$$

we have, by [8, Theorem 7.2.], $h \perp H(Q') \xrightarrow{\sim} h \perp H(Q)$. Thus $q' \xrightarrow{\sim} q$.

Using this proposition we prove

Theorem 3.2. *Let R be a ring of dimension one with finite normalisation and in which 2 is invertible. Let q be a quadratic space over* $R[T,T^{-1}]$ *with Witt index* ≥ 2. *Then q is cancellative.*

Proof. We use the notation of Section 2. It is enough to prove that if q' is a quadratic space over A such that $q \perp h \xrightarrow{\sim} q' \perp h$, then $q \xrightarrow{\sim} q'$. Since Witt index $q \geq 2$ there exists a rank-1 projective A-module P and a quadratic space q_0 over A such that $q=q_0 \perp H(P) \perp h$. Since $\bar{R}$ is a product of Dedekind domains, by Proposition 3.1 we have an isometry

$$q' \otimes_A \bar{A} \xrightarrow[\sim]{\varphi} (q_0 \perp H(P) \perp h) \otimes_A \bar{A}.$$

Since $\dim A/\mathfrak{C}=1$, by [8, Theorem 7.2] we have an isometry

$$q' \otimes_A A/\mathfrak{C} \xrightarrow[\sim]{\psi} (q_0 \perp H(P) \perp h) \otimes_A A/\mathfrak{C}$$

thus $\psi^* \varphi^{*-1} \in O_{\bar{A}/\mathfrak{C}}(q_0 \perp H(P) \perp h)$ where '$*$' denotes the extension to $\bar{A}/\mathfrak{C}$. Since

$\bar{R}/\mathfrak{C}$ is a product of fields modulo its radical, by Lemma 1.3 we have $\psi^*\varphi^{*-1}=\tau\eta$ for some $\tau\in O_{\bar{A}/\mathfrak{C}}(h)$, $\eta\in EO_{\bar{A}/\mathfrak{C}}(q_0\perp H(P),h)$. Since $\bar{A}\to\bar{A}/\mathfrak{C}$ is a surjection, we can lift η to $EO_{\bar{A}}(q_0\perp H(P),h)$ and alter φ suitably to assume that $\varphi^*\psi^{*-1}=\tau$. Using the Cartesian square (2) we get a rank-2 quadratic space q_2 over A which corresponds to the triple (h,τ,h). Then, since the triples (q',Id,q') and $(q_0\perp H(P)\perp h$, $\mathrm{Id}\perp\tau$, $q_0\perp H(P)\perp h)$ are equivalent, we get $q'\xrightarrow{\sim}q_0\perp H(P)\perp q_2$. To complete the proof we show that $H(P)\perp q_2\xrightarrow{\sim}H(P)\perp h$. Since $\dim R=1$ and q and q' are stably isometric, by [8, Theorem 7.2], $\tilde{q}'\xrightarrow{\sim}\tilde{q}$. Hence we have an isometry $H(\tilde{P})\perp\tilde{q}_2\xrightarrow{\theta}H(\bar{P})\perp h$ over R. Since $\bar{R}$ is a product of Dedekind domains, $\operatorname{Pic}\bar{R}=\operatorname{Pic}\bar{A}$. Also $q_2\otimes_A\bar{A}\xrightarrow{\sim}h$. Thus we have isometries

$$\varphi_1:(H(P)\perp q_2)\otimes_A\bar{A}\xrightarrow{\sim}(H(\tilde{P})\perp\tilde{q}_2)\otimes_R\bar{A}$$

and

$$\varphi_2:(H(P)\perp h)\otimes_A\bar{A}\xrightarrow{\sim}(H(\tilde{P})\perp h)\otimes_R\bar{A}$$

over $\bar{A}$ such that $\tilde{\varphi}_1=\mathrm{Id}=\tilde{\varphi}_2$, and hence we get an isometry

$$\varphi'=\varphi_2^{-1}\theta\varphi_1:(H(P)\perp q_2)\otimes_A\bar{A}\xrightarrow{\sim}(H(P)\perp h)\otimes_A\bar{A}$$

such that $\tilde{\varphi}'=\theta$. Similarly, since $\dim R/\mathfrak{C}=0$, we have an isometry

$$\psi':(H(P)\perp q_2)\otimes_A A/\mathfrak{C}\xrightarrow{\sim}(H(P)\perp h)\otimes_A A/\mathfrak{C}$$

such that $\tilde{\psi}'=\theta$. Then $\varphi'^*\psi'^{*-1}\in O_{\bar{A}/\mathfrak{C}}(H(P)\perp h)$. By Lemma 1.3 there exist $\eta'\in EO_{\bar{A}/\mathfrak{C}}(H(P),h)$ and $\tau'\in O_{\bar{A}/\mathfrak{C}}(h)$ such that $\varphi'^*\psi'^{*-1}=\eta'\tau'$. Hence $\tilde{\eta}'\tilde{\tau}'=\mathrm{Id}$ so that $\tilde{\tau}'\in EO_{\bar{R}/\mathfrak{C}}(H(\tilde{P}),h)$. Thus, by Lemma 1.1, $\tilde{\tau}'=\tau_{u_0^2}$ for some $u_0\in U(\bar{R}/\mathfrak{C})$. Since $\mu_2(\bar{A}/\mathfrak{C})=\mu_2(\bar{R}/\mathfrak{C})$ we have $\det\tau'=\det\tilde{\tau}'=1$ and hence $\tau'=\tau_u$ for some $u\in U(\bar{A}/\mathfrak{C})$ with $\tilde{u}=u_0^2$. We can lift η' to an element of $EO_{\bar{A}}(H(P),h)$ and alter φ' suitably to assume $\varphi'^*\psi'^{*-1}=\tau_u$. Since $q\perp h\xrightarrow{\sim}q'\perp h$ and Witt index $H(P)\perp h\perp h\geq 2$, by [8, Theorem 7.2] we have $(H(P)\perp h)\perp h\xrightarrow{\sim}(H(P)\perp q_2)\perp h$ so that the triples $((H(P)\perp h)\perp h$, $\mathrm{Id}\perp\mathrm{Id}$, $(H(P)\perp h)\perp h)$ and $((H(P)\perp h)\perp h$, $\tau_u\perp\mathrm{Id}$, $(H(P)\perp h)\perp h)$ are equivalent. Hence there exist $\alpha\in O_{\bar{A}}(q)$ and $\beta\in O_{A/\mathfrak{C}}(q)$ such that $\alpha^*\beta^{*-1}=\tau_u\perp\mathrm{Id}$. Thus, applying the spinor norm homomorphism [1, 3.3] we get

$$\mathrm{SN}(\alpha^*)\mathrm{SN}(\beta^{*-1})=\mathrm{SN}(\tau_u\perp\mathrm{Id})=\langle u\rangle\qquad[1,4.4.1].$$

We have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U(\bar{A})/U(\bar{A})^2 & \longrightarrow & \operatorname{Disc}\bar{A} & \xrightarrow{\theta_1} & {}_2\operatorname{Pic}\bar{A} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & U(\bar{R})/U(\bar{R})^2 & \longrightarrow & \operatorname{Disc}\bar{R} & \xrightarrow{\theta_2} & {}_2\operatorname{Pic}\bar{R} & \longrightarrow & 0
\end{array}$$

connected by the natural inclusions. Since $\mathrm{SN}(\alpha^*)=\mathrm{SN}(\alpha)\in\operatorname{Disc}\bar{A}$, there exists $\gamma\in\operatorname{Disc}\bar{R}$ such that $\theta_2(\gamma)=\theta_1(\mathrm{SN}(\alpha))$ and hence there exists $\upsilon\in U(\bar{A})$ such that $\upsilon=\gamma^{-1}\otimes\mathrm{SN}(\alpha)$ modulo $U(\bar{A})^2$. Thus $\mathrm{SN}(\alpha^*)=\upsilon\langle\gamma'\rangle$, where $\gamma'\in U(\bar{R}/\mathfrak{C})$, since Pic

$\bar{A}/\mathfrak{C}=0$. Also $\mathrm{SN}(\beta^{*-1})=\mathrm{SN}(\beta)^{-1}=\langle a\rangle$ for some $a\in U(A/\mathfrak{C})$, since $\mathrm{Pic}\, A/\mathfrak{C}=0$. Thus, over $\bar{A}/\mathfrak{C}$, $\langle u\rangle=\langle v\rangle\langle\gamma'\rangle\langle a\rangle$. Since $\mathrm{Disc}\,\bar{A}/\mathfrak{C}\twoheadrightarrow U(\bar{A}/\mathfrak{C})/U(\bar{A}/\mathfrak{C})^2$, there exists $b\in U(\bar{A}/\mathfrak{C})$ such that $u=v\gamma' ab^2$. Thus $u_0^2=\tilde{u}=\tilde{v}\gamma'\tilde{a}\tilde{b}^2$ and hence $u=(v\tilde{v}^{-1})$ $(a\tilde{a}^{-1})(u_0^2b^2\tilde{b}^{-2})$. By Lemma 1.2, $\mathrm{Id}\perp\tau_{u_0^2b^2\tilde{b}^{-2}}\in EO_{\bar{A}/\mathfrak{C}}(H(P),h)$, and hence can be lifted to $EO_{\bar{A}}(H(P),h)$. Also $v\tilde{v}^{-1}\in U(\bar{A})$ and $a\tilde{a}^{-1}\in U(A/\mathfrak{C})$. Thus we can alter φ' and ψ' suitably to obtain $\varphi'^{*}\psi'^{*-1}=\mathrm{Id}$. Hence $H(P)\perp q_2\twoheadrightarrow H(P)\perp h$ and $q'\twoheadrightarrow q$.

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